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Current, maximum power and optimized efficiency of a Brownian heat engine

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Abstract. A microscopic heat engine is modeled as a Brownian particle in a sawtooth potential (with load) moving through a highly viscous medium driven by the thermal kick it gets from alternately placed hot and cold heat reservoirs. We found closed form expression for the current as a function of the parameters characterizing the model. Depending on the values these model parameters take, the engine is also found to function as a refrigerator. Expressions for the efficiency as well as for the refrigerator performance are also reported. Study of how these quantities depend on the model parameters enabled us in identifying the points in the parameter space where the engine performs with maximum power and with optimized efficiency. The corresponding efficiencies of the engine are then compared with those of the endoreversible and Carnot engines.

PACS. 05.40.Jc Brownian motion – 05.60.-k Transport processes – 05.70.-a Thermodynamics

Now-a-days there is much interest in the study of microscopic engines. One aspect is the need to have microscopic engines in order to utilize energy resources available at the microscopic scale. The other aspect is the trend in miniaturization of devices demanding tiny engines that operate at the same scale. During the past decade or so, study of how motor proteins function has given us a guide as to how man-made molecular engines should be designed and constructed [1,2]. We still have a few theoretical issues about microscopic engines that should be addressed in detail.

One of the important points to be addressed about engines is their energetics. Recently, there are studies on the energetics of different kinds of microscopic engines [3]. Any task such an engine performs, including translocating chemicals through a viscous medium with finite velocity, costs energy. In order to find out how efficient such an engine is, one has to generalize the definition of efficiency. This is what Derényi et al. did in their crucial paper on generalized efficiency [4] which was originally proposed by Jülicher et al. [5]. Another issue of how an engine operates is whether a fastest transporting velocity exists under a given condition. Such fast transport is at the expense of consuming more energy and implying less efficiency. A compromise between fast transport and energy cost may lead to an optimized efficiency. Herńandez et al. recently came up with a unified optimization method for doing exactly this [6].

The aim of this paper is to explore the performance of a microscopic (or Brownian) heat engine under various conditions of practical interest such as maximum power and optimized efficiency.

The idea of a Brownian heat engine working due to nonuniform temperature first came up with the works of Büttiker [7], van Kampen [8], and Landauer [9] while they were involved in exposing the significance of the now influential papers of Landauer on blowtorch effect [10,11]. Millonas studied the kinetics of a heat engine, which he called as "information engine", relating it to the underlying microscopic thermodynamics [12]. Following Büttiker's work, Matsuo and Sasa [13] took a Brownian heat engine as an example to show that it acts as a Carnot engine at quasistatic limit. Derényi and Astumian [14], after analyzing the details of heat flow in a Brownian heat engine, found that its efficiency can, in principle, approach that of a Carnot engine. The same group later used a model Brownian heat engine as an example to apply their newly introduced definition of generalized efficiency and find its implications [4]. Recently, Hondou and Sekimoto [15] claimed that such a heat engine cannot attain Carnot efficiency due to inevitable irreversible heat flow over the temperature boundary.

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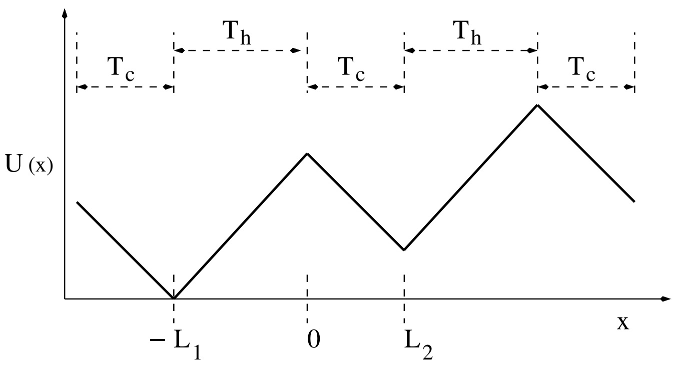

Fig. 1. Plot of the sawtooth potential in the presence of constant external load. The temperature profile is shown above the potential profile.

All these recent works either addressed how a Brownian heat engine behaves in the quasistatic limit or did not deal with its energetics. Instead of limiting ourselves to an issue of purely theoretical interest like quasistatic limit, we would like to take an exactly solvable model from which we can extract information of practical interest. Not only will we be dealing with the quasistatic limit but with the general characteristics of real engines performing tasks either at maximum rate or at optimized efficiency or otherwise. Even though the model is simple, the results we get are of general significance not limited to the model.

The model consists of a Brownian particle moving in a sawtooth potential with an external load where the viscous medium is alternately in contact with hot and cold heat reservoirs along the space (or reaction) coordinate. The shape of a single sawtooth potential, $U_s(x)$, located around $x = 0$ is described by

$$U_s(x) = \begin{cases} U_0\left(\frac{x}{L_1} + 1\right), \text{ if } -L_1 \le x < 0; \\ U_0\left(\frac{-x}{L_2} + 1\right), \text{ if } 0 \le x < L_2. \end{cases}$$

The potential corresponding to the external load is linear, fx, where f is the load. The temperature profile, $T(x)$, within the interval $-L_1 \le x < L_2$ is described by

$$T(x) = \begin{cases} T_h, \text{ if } -L_1 \le x < 0; \\ T_c, \text{ if } 0 \le x < L_2. \end{cases}$$

Both $U_s(x)$ and $T(x)$ are taken to have the same period such that $U_s(x + L) = U_s(x)$ and $T(x + L) = T(x)$ where $L = L_1 + L_2$. Note that the left side of each sawtooth from its barrier top overlaps with the hot region of the medium while the right side overlaps with the cold region. The sawtooth potential with the load, $U(x) = U_s(x) + fx$, and the temperature profile, $T(x)$, are shown in Figure 1.

From practical point of view, the size of such microscopic heat engine is limited by how small in size temperature gradients can be. Since temperature gradients of micron size can be produced [5], the size of such an engine could be as small as few microns.

Due to the presence of the hot and cold regions within each sawtooth potential and the external load, the Brownian particle will generally be driven unidirectionally and attain a steady state current, J, whose magnitude and direction depend upon the quantities characterizing the model. It has been found that the dynamic equation governing the Brownian particle in inhomogeneous media depends on the specific environment to which the particle is exposed [16,17]. For our case, we take the inhomogeneous medium to be highly viscous and its corresponding dynamic equation takes a specific form of Smoluchowski equation first derived by Sancho et al. [18] and later by van Kampen [8] and by Jayannavar and Mahato [19]:

$$\frac{\partial}{\partial t}(P(x,t)) = \frac{\partial}{\partial x}\left[\frac{1}{\gamma(x)}\left(U'(x)P + \frac{\partial}{\partial x}(T(x)P)\right)\right], \quad (1)$$

where $P = P(x, t)$ is the probability density of finding the particle at position x at time t, $U'(x) = dU(x)/dx$, and $\gamma(x)$ is the coefficient of friction at position x. Boltzmann's constant, k_B, is taken to be unity. The constant current at steady state is given by

$$J = -\frac{1}{\gamma(x)}\left[U'(x)P_{ss}(x) + \frac{d}{dx}(T(x)P_{ss}(x))\right], \quad (2)$$

where $P_{ss}(x)$ is the steady state probability density at position x. Using periodic boundary condition, $P_{ss}(x + L) = P_{ss}(x)$, and taking coefficient of friction the same throughout the medium, we get a closed form expression for J for our potential and temperature profiles [20]:

$$J = \frac{-F}{G_1 G_2 + HF}, \quad (3)$$

where F, G_1 and G_2 are

$$F = e^{a-b} - 1,$$
$$G_1 = \frac{L_1}{aT_h}\left(1 - e^{-a}\right) + \frac{L_2}{bT_c}e^{-a}\left(e^b - 1\right),$$
$$G_2 = \frac{\gamma L_1}{a}\left(e^a - 1\right) + \frac{\gamma L_2}{b}e^a\left(1 - e^{-b}\right). \quad (4)$$

On the other hand, H can be put as a sum of three terms: $A + B + C$, where

$$A = \frac{\gamma}{T_h}\left(\frac{L_1}{a}\right)^2 (a + e^{-a} - 1),$$
$$B = \frac{\gamma L_1 L_2}{abT_c}(1 - e^{-a})(e^b - 1),$$
$$C = \frac{\gamma}{T_c}\left(\frac{L_2}{b}\right)^2 (e^b - 1 - b). \quad (5)$$

Note that $a = (U_0 + fL_1)/T_h$ and $b = (U_0 - fL_2)/T_c$ in equations (6) and (7). For the case when there is no load ($f = 0$) and $L_2 = L_1$, the current takes a simple expression given by

$$J = \frac{1}{2\gamma(T_h + T_c)}\left(\frac{U_0}{L_1}\right)^2\left(\frac{1}{e^{\frac{U_0}{T_h}} - 1} - \frac{1}{e^{\frac{U_0}{T_c}} - 1}\right). \quad (6)$$

Note that this current is a resultant of currents to the right and to the left; i.e. $J = J_+ - J_-$.

One can clearly see that the model acts as a heat engine when the current is to the right. The particle undergoes cyclic motion wherein during each cycle it is first in contact with the hot region of width L_1 and then with the cold region of width L_2. We take account of energy flows between the two regions neglecting energy transfer via kinetic energy due to the particle's recrossing of the boundary between the regions [4,14]. As the particle moves through the hot region it absorbs energy, Q_h, which will enable it not only to climb up the potential (U_0+fL_1) but also acquire energy $\gamma v L_1$ (v being the particle's average drift velocity and equal to $J(L_1 + L_2)$) to overcome the region's viscous drag force so that

$$Q_h = U_0 + (\gamma v + f)L_1. \tag{7}$$

On the other hand, the cold region will absorb energy as the particle moves down the potential hill while at the same time lose some energy due to the drag force in the region. Therefore, the net heat, Q_c, absorbed by the cold region will be

$$Q_c = U_0 - (\gamma v + f)L_2. \tag{8}$$

The net work, W, done by the engine in one cycle will then be the difference between Q_h and Q_c so that

$$W = (\gamma v + f)(L_1 + L_2). \tag{9}$$

Note that the dissipation energy via friction is minimum since the particle's motion, determined globally, is uniform. The term $(\gamma v + f)$ appearing in the above three expressions (Eqs. (7–9)) can then be taken as generalized load (or external forces as in [5]) so that the generalized efficiency, η, as suggested by Derényi et al. [4] will be given by

$$\eta = \frac{W}{Q_h} = \frac{(\gamma v + f)(L_1 + L_2)}{U_0 + (\gamma v + f)L_1}. \tag{10}$$

If, on the other hand, the load is large enough along with appropriately chosen other quantities then the current will be in the reverse direction in which case the load does work of amount $W_L = f(L_1 + L_2)$ in one cycle forcing heat to be extracted from the cold region of amount $Q_c = U_0 - (\gamma v + f)L_2$. Under this condition the engine acts as a refrigerator. This will then lead us to a definition of generalized coefficient of performance (COP) of the refrigerator, P_{ref}, which is given by

$$P_{ref} = \frac{Q_c}{W_L} = \frac{U_0 - (\gamma v + f)L_2}{f(L_1 + L_2)}. \tag{11}$$

The condition at which the current changes its direction is the boundary demarcating the domain of operation of the engine as a refrigerator from that as a heat engine. In general, this condition is satisfied when

$$f = \frac{U_0(T_h - T_c)}{L_1T_c + L_2T_h}. \tag{12}$$

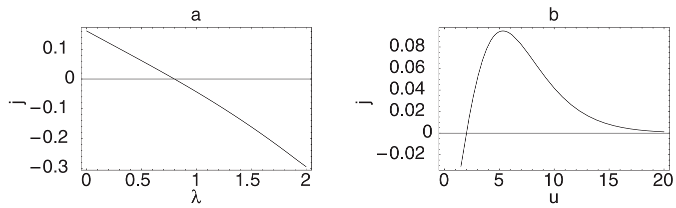

Fig. 2. (a) Plot of j versus λ for $\tau = 1$, $\ell = 2$ and $u = 4$. (b) Plot of j versus u for $\tau = 1$, $\ell = 2$ and $\lambda = 0.4$.

It is worth noting that the magnitude of the load at this point of zero current is exactly equal to what is usually called the stall force for molecular engines [5,21]. When we evaluate the expressions for both η and P_{ref} as we approach this boundary, we analytically find that they are exactly equal to the values for efficiency of the Carnot engine and for COP of the Carnot refrigerator, respectively: $\lim_{J\to 0^+} \eta = \frac{T_h - T_c}{T_h}$ and $\lim_{J\to 0^-} P_{ref} = \frac{T_c}{T_h - T_c}$. Hence, this boundary at which current is zero corresponds to the quasistatic limit be it from the heat engine side or from the refrigerator side. Only for this quasistatic limit ($v \to 0$) will the heat engine operate reversibly, i.e., the entropy production is zero. This result agrees with the comment given by Jülicher et al. [5]. However, the derivation of equation (10) of Derényi et al. [4] and its implications do not hold for a heat engine delivering a finite task in a finite time.

We next explore how current, efficiency and COP of the model depend on some of the quantities characterizing it. In general, the quantities characterizing the model are U_0, L_1, L_2, f, T_c and T_h. We scale U_0, L_2, T_h and f such that $u = U_0/T_c$, $\ell = L_2/L_1$, $\tau = (T_h/T_c) - 1$, and $\lambda = fL_1/T_c$. Hence, we have four parameters u, ℓ, τ and λ characterizing the model for a given T_c and L_1. We also scale current such that $j = J/J_0$, where $J_0 = T_c/(\gamma L_1^2)$. Figure 2a is a plot of the scaled current, j, versus scaled load, λ. It shows that the engine works as a heat engine when the load value is less than the stall force ($\lambda = 0.8$ in this case) and as a refrigerator when the load is larger than this value. One can also see these two domains of the engine by plotting j as a function of u as shown in Figure 2b. Note that when the model works as a heat engine there is a finite u at which the current is maximum. This corresponds to the point in the parameter space at which the engine operates with maximum power.

When we plot the efficiency, η, versus λ within the domain where the model works as a heat engine, we find that it increases linearly with increase in λ attaining its maximum value (Carnot efficiency) at the stall force (see Fig. 3a). On the other hand, Figure 3b shows plot of P_{ref} versus λ which starts from its maximum value (COP of Carnot refrigerator) at the stall force and decreases fast as we increase the load.

Let us now compare the efficiency of our engine with that of the so-called endoreversible engine when both operate with maximum power. Curzon and Ahlborn [22,23] took an endoreversible engine that exchanges heat linearly at finite rate with the two heat reservoirs and found

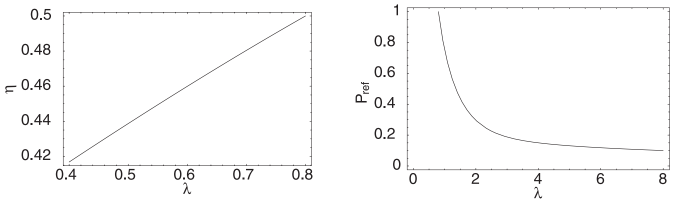

Fig. 3. (a) Plot of η versus λ for $\tau = 1$, $\ell = 2$ and $u = 4$. (b) Plot of P_{ref} versus λ for $\tau = 1$, $\ell = 2$ and $u = 4$.

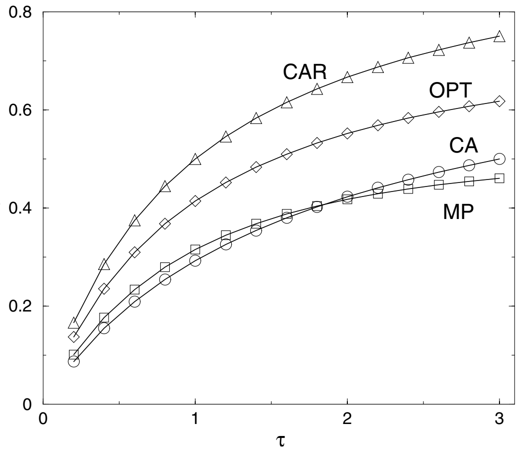

Fig. 4. Plots of η_{CA}, η_{MP}, η_{OPT} and η_{CAR} versus τ, where the model engine is put to function at $f = 0$ and $\ell = 2$ while u is fixed depending on whether it is working at either maximum power or optimized efficiency.

its efficiency at maximum power, η_{CA}, to be equal to $1 - \sqrt{(T_c/T_h)}$. Figure 4 gives plots comparing η_{CA} with that of the efficiency of our engine when it also operates with maximum power, η_{MP}, for different values of τ. The plots show that the two results are reasonably close to each other for small τ with η_{CA} being slightly smaller than η_{MP}. The values of the two efficiencies coincide at a finite value of τ and η_{CA} progressively gets larger than that of η_{MP} for higher values of τ and ultimately approaches that of Carnot efficiency, i.e. 100 percent as $\tau \to \infty$. For the particular asymmetric ($\ell = 2$) sawtooth potential considered here, we have semi-analytically found that the *maximum* value η_{MP} can take (i.e. when $\tau \to \infty$) is very close to 57.75 percent. From the behavior of the two efficiencies, we can conclude that the expression for η_{CA} is of limited significance as it is derived from simple assumption while η_{MP} is general and works for the entire range in the allowed parameter space.

We would further like to compare Carnot efficiency, η_{CAR}, with what is called optimized efficiency, η_{OPT}, by Herńandez et al. [6]. This optimized efficiency is the efficiency at the point of operation of an engine where competition between energy cost and fast transport is compromised. We briefly summarize the method Herńandez et al. [6] used to get η_{OPT}. When the engine operates with finite time the amount of work, W, it delivers per cycle is such that it lies between the maximum, W_{max}, and minimum, W_{min}, amount of work that can be extracted from the engine: $W_{min} \leq W \leq W_{max}$. They defined two quantities: effective work $W_e = W - W_{min}$ and noneffective work $W_{ne} = W_{max} - W$ and introduced a function, Ω, to be optimized to be equal to the difference between these quantities. For our engine (since $W_{min} = 0$) this function is given by

$$\Omega = 2W - \left(\frac{\tau}{1+\tau}\right) Q_h. \tag{13}$$

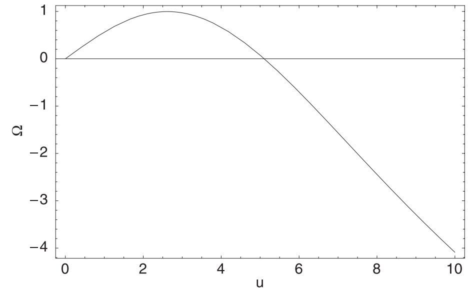

Fig. 5. Plot of Ω versus u for $f = 0$, $\ell = 2$ and $\tau = 1$.

Figure 5 shows plot of Ω versus u in which the function has indeed an optimum at a finite value of u. The efficiency of the engine when it operates at this particular point in the parameter space is what we call as optimized efficiency, η_{OPT}. For the sake of comparison, we plot η_{OPT} along with the corresponding Carnot efficiency, η_{CAR}, in the same figure (Fig. 4) as we did for the other efficiencies. The plots clearly show that η_{OPT} always lies between the η_{CAR} and η_{MP}. One can therefore say that the operation of the engine at optimized efficiency is indeed a compromise between fast transport and energy cost. Lastly, we have also semi-analytically found that the *maximum* value η_{OPT} can take (for the particular case considered) is very close to 75.73 percent and occurs when $\tau \to \infty$.

In this paper, we found a closed form expression for the steady state current of a Brownian heat engine. This enabled us to directly quantify the recently introduced generalized efficiency [4] as well as the generalized COP we introduced. We verified that both these quantities took the correct values at the quasistatic limits. The values we get for the efficiency of the engine at various points in the parameter space where the engine operates with maximum power are new results generally different from those values one gets by using the finite-rate linear heat exchange assumption of Curzon and Ahlborn [22]. Optimization criteria [6] has been implemented to determine the set of points in the parameter space at which efficiency is optimized under each condition and its corresponding value found. Upper limits for the efficiencies when the engine operates with maximum power and with optimized efficiency are reported for a specific asymmetric sawtooth potential we studied.

Lastly, two points need to be mentioned in connection with the validity of this work. Firstly, we have limited our consideration to the overdamped motion of the Brownian particle. It is worth to extend the study to the underdamped regime. Blanter and Büttiker have already investigated how a non-uniform temperature can produce a steady, directed current of particles in a periodic potential in the underdamped regime [24]. Secondly, we have neglected energy transfer between the hot and cold reservoirs via kinetic energy due to the particle's recrossing of the boundaries. This is in contrast to the argument presented by Honduo and Sekimoto [15]. Our opinion is that the specific environment in which the Brownian particle is found determines the validity of neglecting such energy transfer. In any case, this issue is controversial yet to be resolved.

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